AMPLE THOUGHTS

DANIEL PALACÍN AND FRANK O. WAGNER

Abstract. Non-n-ampleness as defined by Pillay [20] and Evans [6] is preserved under analysability. Generalizing this to a more general notion of Σ-ampleness, this gives an immediate proof for all simple theories of a weakened version of the Canonical Base Property (CBP) proven by Chatzidakis [5] for types of finite SU-rank. This is then applied to the special case of groups.

1. Introduction

Recall that a partial type π over a set A in a simple theory is *one-based* if for any tuple $\bar{a}$ of realizations of π and any $B \supseteq A$ the canonical base $\mathrm{Cb}(\bar{a}/B)$ is contained in the bounded closure $\mathrm{bdd}(\bar{a}A)$. In other words, forking dependence is either trivial or behaves as in modules: Any two sets are independent over the intersection of their bounded closures. One-basedness implies that the forking geometry is particularly well-behaved; for instance one-based groups are bounded-by-abelian-by-bounded. The principal result in [26] is that one-basedness is preserved under analyses (i.e. iterative approximations by some other types): a type analysable in one-based types is itself one-based. This generalized earlier results of Hrushovski [12] and Chatzidakis [5]. One-basedness is the first level in a hierarchy of possible geometric behaviour of forking independence first defined by Pillay [20] and slightly modified by Evans [6], n-ampleness, modelled on the behavior of flags in n-space. Not 1-ample means one-based; not 2-ample is equivalent to a notion previously introduced by Hrushovski [13], CM-triviality. Fields

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are n-ample for all $n < \omega$, as is the non-abelian free group [17]. In [20] Pillay defines n-ampleness locally for a single type and shows that a superstable theory of finite Lascar rank is non n-ample if and only if all its types of rank 1 are; his proof implies that in such a theory, a type analysable in non n-ample types is again non n-ample.

We shall give a definition of n-ampleness for invariant families of partial types, and generalize Pillay's result to arbitrary simple theories. Note that for $n = 1$ this gives an alternative proof of the main result in [26]. Since for types of infinite rank the algebraic (bounded) closure used in the definition is not necessarily appropriate (for a regular type p one might, for instance, replace it by p-closure), we also generalize the notion to Σ-closure for some $\emptyset$-invariant collection of partial types (thought of as small), giving rise to the notion of n-Σ-ample. This may for instance be applied to consider ampleness modulo types of finite SU-rank, or modulo supersimple types. Readers not interested in this additional generality are invited to simply replace Σ-closure by bounded closure. However, this will only marginally shorten the proofs. As an immediate Corollary of the more general version, we shall derive a weakened version of the Canonical Base Property CBP [23] shown by Chatzidakis [5], where analysability replaces internality in the definition. We also give a version appropriate for supersimple theories. Finally, we deduce that in a simple theory with enough regular types, a hyperdefinable group modulo its approximate centre is analysable in the family of non one-based regular types; the group modulo a normal nilpotent subgroup is almost internal in that family. This can be thought of as a general version of the properties of one-based groups mentioned above.

Our notation is standard and follows [25]. Throughout the paper, the ambient theory will be simple, and we shall be working in $\mathfrak{M}^{heq}$, where $\mathfrak{M}$ is a sufficiently saturated model of the ambient theory. Thus tuples are tuples of hyperimaginaries, and $\mathrm{dcl} = \mathrm{dcl}^{heq}$.

2. Internality and analysability

For the rest of the paper Σ will be an $\emptyset$-invariant family of partial types. Recall first the definitions of internality, analysability, foreignness and orthogonality.

Definition 2.1. Let π be a partial type over A. Then π is

- (*almost*) Σ-*internal* if for every realization a of π there is $B \mathop{\smile\!\!\!\!\!\mid}_A a$ and a tuple $\bar b$ of realizations of types in Σ based on B, such that $a \in \mathrm{dcl}(B\bar b)$ (or $a \in \mathrm{bdd}(B\bar b)$, respectively).
- Σ-*analysable* if for any realization a of π there are $(a_i : i < \alpha) \in \mathrm{dcl}(Aa)$ such that $\mathrm{tp}(a_i/A, a_j : j < i)$ is Σ-internal for all $i < \alpha$, and $a \in \mathrm{bdd}(A, a_i : i < \alpha)$.

A type $\mathrm{tp}(a/A)$ is *foreign* to Σ if $a \mathop{\smile\!\!\!\!\!\mid}_{AB} \bar b$ for all $B \mathop{\smile\!\!\!\!\!\mid}_A a$ and $\bar b$ realizing types in Σ over B.
Finally, $p \in S(A)$ is *orthogonal* to $q \in S(B)$ if for all $C \supseteq AB$, $a \models p$, and $b \models q$ with $a \mathop{\smile\!\!\!\!\!\mid}_A C$ and $b \mathop{\smile\!\!\!\!\!\mid}_B C$ we have $a \mathop{\smile\!\!\!\!\!\mid}_C b$.

So p is foreign to Σ if p is orthogonal to all completions of partial types in Σ, over all possible parameter sets.

The following lemmas and their corollaries are folklore, but we add some precision about non-orthogonality.

Lemma 2.2. *Suppose $a \mathop{\smile\!\!\!\!\!\mid} b$ and $a \not\mathop{\smile\!\!\!\!\!\mid}_b c$. Let $(b_i : i < \omega)$ be an indiscernible sequence in $\mathrm{tp}(b)$ and put $p_b = \mathrm{tp}(c/b)$. Then p_{b_i} is non-orthogonal to p_{b_j} for all $i, j < \omega$.*

Proof: We prolong the sequence to have length α. As $a \mathop{\smile\!\!\!\!\!\mid} b$ and $(b_i : i < \alpha)$ is indiscernible, by [25, Theorem 2.5.4] we may assume $ab \equiv ab_i$ for all $i < \alpha$ and $a \mathop{\smile\!\!\!\!\!\mid} (b_i : i < \alpha)$. Let $B = (b_i : i < \omega)$, so $(b_i : \omega \le i < \alpha)$ is independent over B and $a \mathop{\smile\!\!\!\!\!\mid} B$. Choose c_i with $b_i c_i \equiv_a bc$ and

$$c_i \mathop{\smile\!\!\!\!\!\mid}_{ab_i} (b_j : j < \alpha)$$

for all $\omega \le i < \alpha$. Then $ab_ic_i \mathop{\smile\!\!\!\!\!\mid}_B (b_j : j \neq i)$ for all $\omega \le i < \alpha$. By indiscernability, if p_{b_i} were orthogonal to p_{b_j} for some $i \neq j$, then they would be orthogonal for all $i \neq j$. As $c_i \mathop{\smile\!\!\!\!\!\mid}_{b_i} (b_j : j \neq i)$, the sequence $(b_ic_i : \omega \le i < \alpha)$ would be independent over B. However, $a \not\mathop{\smile\!\!\!\!\!\mid}_B b_ic_i$ for all $\omega \le i < \alpha$, contradicting the boundedness of weight of $\mathrm{tp}(a/B)$. $\square$

Lemma 2.3. *Suppose $a \mathop{\smile\!\!\!\!\!\mid} b$ and $a' = \mathrm{Cb}(bc/a)$. Let $\mathcal{P}$ be the family of $\mathrm{bdd}(\emptyset)$-conjugates of $\mathrm{tp}(c/b)$ non-orthogonal to $\mathrm{tp}(c/b)$. Then $a' \in \mathrm{bdd}(a)$ is $\mathcal{P}$-internal and $\mathrm{bdd}(ab) \cap \mathrm{bdd}(bc) \subseteq \mathrm{bdd}(a'b)$.*

Proof: If $a \mathop{\smile\!\!\!\!\!\mid} bc$ then $a' \in \mathrm{bdd}(\emptyset)$ and $\mathrm{bdd}(ab) \cap \mathrm{bdd}(bc) = \mathrm{bdd}(b)$, so there is nothing to show. Assume $a \not\mathop{\smile\!\!\!\!\!\mid}_b c$. Clearly $a' \in \mathrm{bdd}(a)$; as $bc \mathop{\smile\!\!\!\!\!\mid}_{a'} a$ we get $c \mathop{\smile\!\!\!\!\!\mid}_{a'b} a$ and hence $\mathrm{bdd}(ab) \cap \mathrm{bdd}(bc) \subseteq \mathrm{bdd}(a'b)$. Let $(b_ic_i : i < \omega)$ be a Morley sequence in $\mathrm{Lstp}(bc/a)$ with $b_0c_0 = bc$. Then $a' \in \mathrm{dcl}(b_ic_i : i < \omega)$; since $b \mathop{\smile\!\!\!\!\!\mid} a$ we get $(b_i : i < \omega) \mathop{\smile\!\!\!\!\!\mid} a$, whence

$(b_i : i < \omega) \mathop{\smile\!\!\!\!\!\!|}\; a'$. So a' is internal in $\{\mathrm{tp}(c_i/b_i) : i < \omega\}$. Finally, $\mathrm{tp}(c_i/b_i)$ is non-orthogonal to $\mathrm{tp}(c/b)$ for all $i < \omega$ by Lemma 2.2. $\square$

Corollary 2.4. *If $a \mathop{\smile\!\!\!\!\!\!|}\; b$ and $\mathrm{tp}(c/b)$ is (almost) Σ-internal, then $\mathrm{Cb}(bc/a)$ is (almost) Σ-internal. The same statement holds with* analysable *instead of* internal.

Proof: Let $d \mathop{\smile\!\!\!\!\!\!|}_b c$ and $\bar e$ realize partial types in Σ over bd such that $c \in \mathrm{dcl}(bd\bar e)$ (or $c \in \mathrm{bdd}(bd\bar e)$, respectively). We may take $d\bar e \mathop{\smile\!\!\!\!\!\!|}_{bc} a$. Then $d \mathop{\smile\!\!\!\!\!\!|}_b ac$, whence $a \mathop{\smile\!\!\!\!\!\!|}\; bd$. So $\mathrm{Cb}(bd\bar e/a)$ is Σ-internal by Lemma 2.3. But $a \mathop{\smile\!\!\!\!\!\!|}_{bc} d\bar e$ and $c \in \mathrm{dcl}(bd\bar e)$ implies $\mathrm{Cb}(bc/a) \in \mathrm{dcl}(\mathrm{Cb}(bd\bar e/a))$; similarly $c \in \mathrm{bdd}(bd\bar e)$ implies $\mathrm{Cb}(bc/a) \in \mathrm{bdd}(\mathrm{Cb}(bd\bar e/a))$.

The proof for Σ-analysability is analogous. $\square$

Definition 2.5. Two partial types π_1 and π_2 are *perpendicular*, denoted $\pi_1 \perp\!\!\!\perp \pi_2$, if for any set A containing their domains and any tuple $\bar a_i \models \pi_i$ for $i = 1, 2$ we have $\bar a_1 \mathop{\smile\!\!\!\!\!\!|}_A \bar a_2$.

For instance, orthogonal types of rank 1 are perpendicular.

Corollary 2.6. *Suppose $a \mathop{\smile\!\!\!\!\!\!|}\; b$, and $a_0 \in \mathrm{bdd}(ab)$ is (almost) Π-internal over b for some b-invariant family Π of partial types. Let Π' be the family of $\mathrm{bdd}(\emptyset)$-conjugates π' of partial types $\pi \in \Pi$ with $\pi' \not\perp\!\!\!\perp \pi$. Then there is (almost) Π'-internal $a_1 \in \mathrm{bdd}(a)$ with $a_0 \in \mathrm{bdd}(a_1 b)$. The same statement holds with* analysable *instead of* internal.

Proof: If $\mathrm{tp}(a_0/b)$ is Π-internal, there is $c \mathop{\smile\!\!\!\!\!\!|}_b a_0$ and $\bar e$ realizing partial types in Π over bc such that $a_0 \in \mathrm{dcl}(bc\bar e)$; we choose them with $c\bar e \mathop{\smile\!\!\!\!\!\!|}_{ba_0} a$. So $c \mathop{\smile\!\!\!\!\!\!|}_b a$, whence $a \mathop{\smile\!\!\!\!\!\!|}\; bc$. Furthermore, we may assume that $e \not\mathop{\smile\!\!\!\!\!\!|}_{bc} a$ for all $e \in \bar e$, since otherwise $ec \mathop{\smile\!\!\!\!\!\!|}_b a_0$ and we may just include e in c, reducing the length of $\bar e$. Now $a_0 \in \mathrm{bdd}(abc) \cap \mathrm{bdd}(bc\bar e)$, so by Lemmas 2.2 and 2.3 there is Π'-internal $a_1 \in \mathrm{bdd}(a)$ with $a_0 \in \mathrm{bdd}(bca_1)$. Since $a \mathop{\smile\!\!\!\!\!\!|}_b c$ implies $a_0 \mathop{\smile\!\!\!\!\!\!|}_{a_1 b} c$, we get $a_0 \in \mathrm{bdd}(a_1 b)$.

For the almost internal case, we replace definable by bounded closure; for the analysability statement we iterate, adding a_1 to the parameters. $\square$

To finish this section, a decomposition lemma for almost internality.

Lemma 2.7. *Let $\Sigma = \bigcup_{i<\alpha} \Sigma_i$, where $(\Sigma_i : i < \alpha)$ is a collection of pairwise perpendicular $\emptyset$-invariant families of partial types. If $\mathrm{tp}(a/A)$ is almost Σ-internal, then there are $(a_i : i < \alpha)$ interbounded over A with a such that $\mathrm{tp}(a_i/A)$ is Σ_i-internal for $i < \alpha$.*

Clearly, if a is a finite imaginary tuple, we only need finitely many a_i.

Proof: By assumption there is $B \mathop{\smile\!\!\!\!\!\!\,|}_A a$ and some tuples $(b_i : i < \alpha)$ such that b_i realizes partial types in Σ_i over B, with $a \in \mathrm{bdd}(B, b_i : i < \alpha)$. Let $a_i = \mathrm{Cb}(Bb_i/Aa)$. Then $a_i \in \mathrm{bdd}(Aa)$ and $\mathrm{tp}(a_i/A)$ is Σ_i-internal by Corollary 2.4.

Put $\bar a = (a_i : i < \alpha)$. Then $a \mathop{\smile\!\!\!\!\!\!\,|}_{Aa_i} Bb_i$ implies $a \mathop{\smile\!\!\!\!\!\!\,|}_{B\bar a} b_i$; since $b_i \mathop{\smile\!\!\!\!\!\!\,|}_{Ba} (b_j : j \neq i)$ by perpendicularity we obtain $b_i \mathop{\smile\!\!\!\!\!\!\,|}_{B\bar a} (a, b_j : j \neq i)$ for all $i < \alpha$. Hence $(a, b_i : i < \alpha)$ is independent over $B\bar a$, and in particular

$$a \mathop{\smile\!\!\!\!\!\!\,|}_{B\bar a} (b_i : i < \alpha).$$

Since $a \in \mathrm{bdd}(B, b_i : i < \alpha)$ we get $a \in \mathrm{bdd}(B\bar a)$; as $a \mathop{\smile\!\!\!\!\!\!\,|}_A B$ implies $a \mathop{\smile\!\!\!\!\!\!\,|}_{A\bar a} B$ we obtain $a \in \mathrm{bdd}(A\bar a)$. $\square$

3. Σ-closure, Σ_1-closure and a theory of levels

In his proof of Vaught's conjecture for superstable theories of finite rank [3], Buechler defines the first level $\ell_1(a)$ of an element a of finite Lascar rank as the set of all $b \in \mathrm{acl}^{eq}(a)$ internal in the family of all types of Lascar rank one; higher levels are defined inductively by $\ell_{n+1}(a) = \ell_1(a/\ell_n(a))$. The notion has been studied by Prerna Bihani Juhlin in her thesis [1] in connection with a reformulation of the canonical base property. We shall generalise the notion to arbitrary simple theories.

Definition 3.1. For an ordinal α the *α-th Σ-level* of a over A is defined inductively by $\ell_0^\Sigma(a/A) = \mathrm{bdd}(A)$, and for $\alpha > 0$

$$\ell_\alpha^\Sigma(a/A) = \{b \in \mathrm{bdd}(aA) : \mathrm{tp}(b/\bigcup_{\beta<\alpha} \ell_\beta(a/A)) \text{ is almost } \Sigma\text{-internal}\}.$$

Finally, we shall write $\ell_\infty^\Sigma(a/A)$ for the set of all hyperimaginaries $b \in \mathrm{bdd}(aA)$ such that $\mathrm{tp}(b/A)$ is Σ-analysable.

Remark 3.2. Clearly, $\mathrm{tp}(a/A)$ is Σ-analysable if and only if $\ell_\infty^\Sigma(a/A) = \mathrm{bdd}(aA)$ if and only if $\ell_\alpha^\Sigma(a/A) = \mathrm{bdd}(aA)$ for some ordinal α, and the minimal such α is the minimal length of a Σ-analysis of a over A.

Lemma 3.3. *If $a \mathop{\smile\!\!\!\!\!\!\,|} b$, then $\ell_\alpha^\Sigma(ab) = \mathrm{bdd}(\ell_\alpha^\Sigma(a), \ell_\alpha^\Sigma(b))$ for any α.*

Proof: Let $c = \ell_\alpha^\Sigma(ab)$. Clearly, $\ell_\alpha^\Sigma(a)\ell_\alpha^\Sigma(b) \subseteq c$. Conversely, put $a_0 = \mathrm{Cb}(bc/a)$. Then $\mathrm{tp}(a_0)$ is internal in the family of $\mathrm{bdd}(\emptyset)$-conjugates of $\mathrm{tp}(c/b)$ by Corollary 2.4; since even $\mathrm{tp}(c)$ is Σ-analysable in α steps, so is $\mathrm{tp}(a_0)$. Thus $a_0 \subseteq \ell_\alpha^\Sigma(a)$. Now $bc \mathop{\smile\!\!\!\!\!\!|}_{a_0} a$ implies

$$c \mathop{\smile\!\!\!\!\!\!|}_{\ell_\alpha^\Sigma(a)b} a,$$

whence $c \subseteq \mathrm{bdd}(\ell_\alpha^\Sigma(a), b)$. By symmetry, $c \subseteq \mathrm{bdd}(\ell_\alpha^\Sigma(b), a)$, that is,

$$\ell_\alpha^\Sigma(ab) \subseteq \mathrm{bdd}(\ell_\alpha^\Sigma(a), b) \cap \mathrm{bdd}(\ell_\alpha^\Sigma(b), a).$$

On the other hand, $a \mathop{\smile\!\!\!\!\!\!|} b$ yields $a \mathop{\smile\!\!\!\!\!\!|}_{\ell_\alpha^\Sigma(a)\ell_\alpha^\Sigma(b)} b$. Thus,

$$\mathrm{bdd}(\ell_\alpha^\Sigma(a), b) \cap \mathrm{bdd}(\ell_\alpha^\Sigma(b), a) = \mathrm{bdd}(\ell_\alpha^\Sigma(a), \ell_\alpha^\Sigma(b)),$$

whence the result. $\square$

Corollary 3.4. *If $(a_i : i \in I)$ is an $\emptyset$-independent sequence, then $\ell_\alpha^\Sigma(a_i : i \in I) = \mathrm{bdd}(\ell_\alpha^\Sigma(a_i) : i \in I)$.*

Proof: Let $c = \ell_\alpha^\Sigma(a_i : i \in I)$ and set $b_J = \mathrm{Cb}(c/a_i : i \in J)$ for each finite $J \subseteq I$. Note for each finite $J \subseteq I$ that $\mathrm{tp}(b_J)$ is Σ-analysable in α steps. Thus $b_J \subseteq \ell_\alpha^\Sigma(a_i : i \in J)$. On the other hand,

$$\ell_\alpha^\Sigma(a_i : i \in J) = \mathrm{bdd}(\ell_\alpha^\Sigma(a_i) : i \in J)$$

by Lemma 3.3 and induction, since $J \subseteq I$ is finite. Therefore

$$c \mathop{\smile\!\!\!\!\!\!|}_{(\ell_\alpha^\Sigma(a_i) : i \in I)} (a_i : i \in I)$$

by the finite character of forking, whence $c \subseteq \mathrm{bdd}(\ell_\alpha^\Sigma(a_i) : i \in I)$. $\square$

We shall see that the first level governs domination-equivalence.

Definition 3.5. An element a Σ-*dominates* an element b over A, denoted $a \trianglerighteq_A^\Sigma b$, if for all c such that $\mathrm{tp}(c/A)$ is Σ-analysable, $a \mathop{\smile\!\!\!\!\!\!|}_A c$ implies $b \mathop{\smile\!\!\!\!\!\!|}_A c$. Two elements a and b are Σ-*domination-equivalent* over A, denoted $a \sqsubseteq\!\!\!\!\sqsupseteq_A^\Sigma b$, if $a \trianglerighteq_A^\Sigma b$ and $b \trianglerighteq_A^\Sigma a$. If Σ is the set of all types, it is omitted.

The following generalizes a theorem by Buechler [3, Proposition 3.1] for finite Lascar rank.

Theorem 3.6. *Let Σ' be an $\emptyset$-invariant family of partial types.*

1. *a and $\ell_1^\Sigma(a/A)$ are Σ-domination-equivalent over A.*
2. *If $\mathrm{tp}(a/A)$ is Σ-analysable, then a and $\ell_1^\Sigma(a/A)$ are domination-equivalent over A.*
3. *If $\mathrm{tp}(a/A)$ is $\Sigma \cup \Sigma'$-analysable and foreign to Σ', then a and $\ell_1^\Sigma(a/A)$ are domination-equivalent over A.*

Proof: (1) Since $\ell_1^\Sigma(a/A) \in \mathrm{bdd}(Aa)$, clearly a dominates (and Σ-dominates) $\ell_1^\Sigma(a/A)$ over A.

For the converse, suppose $\mathrm{tp}(b/A)$ is Σ-analysable and $b \not\perp_A a$. Consider a sequence $(b_i : i < \alpha)$ in $\mathrm{bdd}(Ab)$ such that $\mathrm{tp}(b_i/A, b_j : j < i)$ is Σ-internal for all $i < \alpha$ and $b \in \mathrm{bdd}(A, b_i : i < \alpha)$. Since $a \not\perp_A b$ there is a minimal $i < \alpha$ such that $a \not\perp_{A,(b_j:j<i)} b_i$. Put

$$a' = \mathrm{Cb}(b_j : j \leq i/Aa) \in \mathrm{bdd}(Aa).$$

Then $\mathrm{tp}(a'/A)$ is Σ-internal by Corollary 2.4, and $a' \subseteq \ell_1^\Sigma(a/A)$. Clearly $a' \not\perp_A (b_j : j \leq i)$, whence $a' \not\perp_A b$ and finally $\ell_1^\Sigma(a/A) \not\perp_A b$. This shows (1).

(2) follows from (3) setting $\Sigma' = \emptyset$.

(3) Suppose $b \not\perp_A a$. We may assume that $b = \mathrm{Cb}(a/Ab)$, so $\mathrm{tp}(b/A)$ is $(\Sigma \cup \Sigma')$-analysable. Thus

$$b \not\perp_A \ell_1^{\Sigma\cup\Sigma'}(a/A)$$

by (1). Now $\mathrm{tp}(\ell_1^{\Sigma\cup\Sigma'}(a/A)/A)$ is foreign to Σ' since $\mathrm{tp}(a/A)$ is; it is hence almost Σ-internal. Therefore $\ell_1^{\Sigma\cup\Sigma'}(a/A) \subseteq \ell_1^\Sigma(a/A)$ and so $b \not\perp_A \ell_1^\Sigma(a/A)$. □

Remark 3.7. If $\mathrm{tp}(a/A)$ is Σ_0-analysable and Σ_1 is a subfamily of Σ_0 such that $\mathrm{tp}(a/A)$ remains Σ_1-analysable, then

$$\ell_1^{\Sigma_1}(a/A) \subseteq \ell_1^{\Sigma_0}(a/A) \subseteq \mathrm{bdd}(aA)$$

and $\ell_1^{\Sigma_1}(a/A)$ and $\ell_1^{\Sigma_0}(a/A)$ are both domination-equivalent to a over A. In fact it would be sufficient to have Σ_1 such that $\mathrm{tp}(\ell_1^{\Sigma_0}(a/A)/A)$ is Σ_1-analysable.

Question 3.8. When is there a minimal (boundedly closed) $a_0 \in \mathrm{bdd}(aA)$ domination-equivalent with a over A?

If T has finite SU-rank, one can take $a_0 \in \mathrm{bdd}(aA) \setminus \mathrm{bdd}(A)$ with $SU(a_0/A)$ minimal possible.

Definition 3.9.
- A type $\mathrm{tp}(a/A)$ is Σ-*flat* if $\ell_1^\Sigma(a/A) = \ell_2^\Sigma(a/A)$. It is A-*flat* if it is Σ-flat for all A-invariant Σ. It is *flat* if for all $B \supseteq A$ every nonforking extension to B is B-flat. A theory T is *flat* if all its types are.
- A type $p \in S(A)$ is A-*ultraflat* if it is almost internal in any A-invariant family of partial types it is non-foreign to. It is *ultraflat* if for any $B \supseteq A$ every nonforking extension to B is B-ultraflat.

Flatness and ultraflatness are clearly preserved under non-forking extensions and non-forking restrictions, and under adding and forgetting parameters.

Remark 3.10. If $\mathrm{tp}(a/A)$ is Σ-flat, then $\ell^\Sigma_\alpha(a/A) = \ell^\Sigma_1(a/A)$ for all $\alpha > 0$. Clearly, ultraflat implies flat.

Example.
- Generic types of fields or definably simple groups interpretable in a simple theory are ultraflat.
- Types of Lascar rank 1 are ultraflat.
- If there is no boundedly closed set between $\mathrm{bdd}(A)$ and $\mathrm{bdd}(aA)$, then $\mathrm{tp}(a/A)$ is A-ultraflat.
- In a small simple theory there are many A-ultraflat types over finite sets A, as the lattice of boundedly closed subsets of $\mathrm{bdd}(aA)$ is scattered for finitary aA.

Next we shall prove that any type internal in a family of Lascar rank one types is also flat.

Lemma 3.11. *It* $\mathrm{tp}(a/A)$ *is flat (ultraflat), then so is* $\mathrm{tp}(a_0/A)$ *for any* $a_0 \in \mathrm{bdd}(Aa)$.

Proof: Consider a set B extending A with $B \mathop{\smile\!\!\!\!\!\!\;|}_A a_0$; we may assume that $B \mathop{\smile\!\!\!\!\!\!\;|}_{Aa_0} a$,whence $B \mathop{\smile\!\!\!\!\!\!\;|}_A a$.

Firstly, the flat case is clear since $\ell^\Sigma_\alpha(a_0/B) = \ell^\Sigma_\alpha(a/B) \cap \mathrm{bdd}(Ba_0)$ for any $\alpha > 0$ and for any B-invariant family Σ. Assume now $\mathrm{tp}(a/A)$ is ultraflat, $a_0 \in \mathrm{bdd}(Aa)$ and $\mathrm{tp}(a_0/B)$ is not foreign to some B-invariant family Σ. Then $\mathrm{tp}(a/B)$ is not foreign to Σ, hence almost Σ-internal, as is $\mathrm{tp}(a_0/B)$. $\square$

Corollary 3.12. *If* $\mathrm{tp}(a/A)$ *is almost internal in a family of types of Lascar rank one, then it is flat.*

Proof: Assume there is some $B \mathop{\smile\!\!\!\!\!\!\;|}_A a$ and some tuple $\bar b$ of realizations of types of Lascar rank one over B such that $a \subseteq \mathrm{bdd}(B\bar b)$. We may assume $\bar b$ is an independent sequence over B since all its elements have SU-rank one. Hence $\bar b$ is an independent sequence over any $C \supseteq B$ with $C \mathop{\smile\!\!\!\!\!\!\;|}_B \bar b$, so $\mathrm{tp}(\bar b/B)$ is flat by Corollary 3.4. Thus, $\mathrm{tp}(a/B)$ is flat by Lemma 3.11, and so is $\mathrm{tp}(a/A)$. $\square$

Question 3.13. Is every (finitary) type in a small simple theory non-orthogonal to a flat type?

Question 3.14. Is every type in a supersimple theory non-orthogonal to a flat type?

Problem 3.15. Construct a flat type which is not ultraflat.

We shall now recall the definitions and properties of Σ-closure from [24, Section 4.0] in the stable and [25, Section 3.5] in the simple case (where it is called P-closure: our Σ corresponds to the collection of all P-analysable types which are co-foreign to P). Buechler and Hoover [2, Definition 1.2] redefine such a closure operator in the context of superstable theories and reprove some of the properties [2, Lemma 2.5].

Definition 3.16. The Σ-*closure* $\Sigma\mathrm{cl}(A)$ of a set A is the collection of all hyperimaginaries a such that $\mathrm{tp}(a/A)$ is Σ-analysable.

Remark 3.17. We think of partial types in Σ as small. We always have $\mathrm{bdd}(A) \subseteq \Sigma\mathrm{cl}(A)$; equality holds if Σ is the family of all bounded types. Other useful examples for Σ are the family of all types of SU-rank $< \omega^\alpha$ for some ordinal α, the family of all supersimple types in a properly simple theory, or the family of p-simple types of p-weight 0 for some regular type p, giving rise to Hrushovski's p-closure [10].

Fact 3.18. *The following are equivalent:*

1. $\mathrm{tp}(a/A)$ *is foreign to* Σ.
2. $a \mathop{\smile\!\!\!\!\!\!\mid}_A \Sigma\mathrm{cl}(A)$.
3. $a \mathop{\smile\!\!\!\!\!\!\mid}_A \mathrm{dcl}(aA) \cap \Sigma\mathrm{cl}(A)$.
4. $\mathrm{dcl}(aA) \cap \Sigma\mathrm{cl}(A) \subseteq \mathrm{bdd}(A)$.

Proof: The equivalence of (1), (2) and (3) is [25, Lemma 3.5.3]; the equivalence (3) $\Leftrightarrow$ (4) is obvious. $\square$

Unless it equals bounded closure, Σ-closure has the size of the monster model and thus violates the usual conventions. The equivalence (2) $\Leftrightarrow$ (3) can be used to cut it down to some small part.

Fact 3.19. *Suppose* $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$. *Then* $\Sigma\mathrm{cl}(A) \mathop{\smile\!\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(B)} \Sigma\mathrm{cl}(C)$. *More precisely, for any* $A_0 \subseteq \Sigma\mathrm{cl}(A)$ *we have* $A_0 \mathop{\smile\!\!\!\!\!\!\mid}_{B_0} \Sigma\mathrm{cl}(C)$, *where* $B_0 = \mathrm{dcl}(A_0B) \cap \Sigma\mathrm{cl}(B)$. *In particular,* $\Sigma\mathrm{cl}(AB) \cap \Sigma\mathrm{cl}(BC) = \Sigma\mathrm{cl}(B)$.

Proof: This is [25, Lemma 3.5.5]; the second clause follows from Fact 3.18. $\square$

Lemma 3.20. *Suppose* $C \subseteq A \cap B \cap D$ *and* $AB \mathop{\smile\!\!\!\!\!\!\mid}_C D$.

1. *If* $\Sigma\mathrm{cl}(A) \cap \Sigma\mathrm{cl}(B) = \Sigma\mathrm{cl}(C)$, *then* $\Sigma\mathrm{cl}(AD) \cap \Sigma\mathrm{cl}(BD) = \Sigma\mathrm{cl}(D)$.

(2) *If* $\mathrm{bdd}(A) \cap \Sigma\mathrm{cl}(B) = \mathrm{bdd}(C)$, *then* $\mathrm{bdd}(AD) \cap \Sigma\mathrm{cl}(BD) = \mathrm{bdd}(D)$.

Proof: (1) This is [25, Lemma 3.5.6], which in turn adapts [18, Fact 2.4].

(2) This is similar to (1). By Fact 15

$$\Sigma\mathrm{cl}(BD) \underset{\Sigma\mathrm{cl}(B)\cap\mathrm{dcl}(AB)}{\perp} AB \ ;$$

since $AD \perp_A AB$ we obtain

$$\mathrm{Cb}(\mathrm{bdd}(AD) \cap \Sigma\mathrm{cl}(BD)/AB) \subseteq \mathrm{bdd}(A) \cap \Sigma\mathrm{cl}(B) = \mathrm{bdd}(C).$$

Hence

$$\mathrm{bdd}(AD) \cap \Sigma\mathrm{cl}(BD) \underset{C}{\perp} AB$$

and by transitivity

$$\mathrm{bdd}(AD) \cap \Sigma\mathrm{cl}(BD) \underset{D}{\perp} ABD,$$

whence the result. □

The following lemma tells us that we can actually find a set C with $\Sigma\mathrm{cl}(A) \cap \Sigma\mathrm{cl}(B) = \Sigma\mathrm{cl}(C)$ as in Lemma 3.20(1), even though the Σ-closures have the size of the monster model.

Lemma 3.21. *Let* $C = \mathrm{bdd}(AB) \cap \Sigma\mathrm{cl}(A) \cap \Sigma\mathrm{cl}(B)$. *Then* $\Sigma\mathrm{cl}(A) \cap \Sigma\mathrm{cl}(B) = \Sigma\mathrm{cl}(C)$.

Proof: Consider $e \in \Sigma\mathrm{cl}(A) \cap \Sigma\mathrm{cl}(B)$ and put $f = \mathrm{Cb}(e/AB)$. Then $e \perp_f AB$; since $\mathrm{tp}(e/A)$ is Σ-analysable, so is $\mathrm{tp}(e/f)$, and $e \in \Sigma\mathrm{cl}(f)$. If I is a Morley sequence in $\mathrm{tp}(e/AB)$, then $f \in \mathrm{dcl}(I)$. However, since e is Σ-analysable over A and over B, so is I, whence f. Hence

$$f \in \mathrm{bdd}(AB) \cap \Sigma\mathrm{cl}(A) \cap \Sigma\mathrm{cl}(B) = C.$$

The result follows. □

However, for considerations such as the canonical base property, one should like to work with the first level of the Σ-closure rather than with the full closure operator.

Definition 3.22. The Σ_1-closure of A is given by

$$\Sigma_1\mathrm{cl}(A) = \ell_1^\Sigma(\Sigma\mathrm{cl}(A)/A) = \{b : \mathrm{tp}(b/A) \text{ is almost } \Sigma\text{-internal}\}.$$

Unfortunately, unless $\mathrm{tp}(\Sigma\mathrm{cl}(A)/A)$ is Σ-flat, Σ_1-closure is not a closure operator, as $\Sigma_1\mathrm{cl}(\Sigma_1\mathrm{cl}(A)) \supset \Sigma_1\mathrm{cl}(A)$.

Lemma 3.23. *Suppose* $A \mathop{\smile\!\!\!\!\!\!\perp}_B C$ *with* $B \subseteq A \cap C$. *Then*

$$\Sigma_1\mathrm{cl}(A) \mathop{\smile\!\!\!\!\!\!\perp}_{\Sigma_1\mathrm{cl}(B)} C.$$

More precisely, $\Sigma_1\mathrm{cl}(A) \mathop{\smile\!\!\!\!\!\!\perp}_{\Sigma_1\mathrm{cl}(B)\cap\mathrm{bdd}(C)} C$.

Proof: Consider $a \in \Sigma_1\mathrm{cl}(A)$ and put $c = \mathrm{Cb}(Aa/C)$. Then $\mathrm{tp}(c/B)$ is almost Σ-internal by Corollary 2.4, and $c \in \mathrm{bdd}(C) \cap \Sigma_1\mathrm{cl}(B)$. □

Question 3.24. If $A \mathop{\smile\!\!\!\!\!\!\perp}_B C$, is $\Sigma_1\mathrm{cl}(A) \mathop{\smile\!\!\!\!\!\!\perp}_{\Sigma_1\mathrm{cl}(B)} \Sigma_1\mathrm{cl}(C)$?

4. Σ-ampleness and weak Σ-ampleness

Let Φ and Σ be $\emptyset$-invariant families of partial types.

Definition 4.1. Φ is n-Σ-*ample* if there are tuples $a_0, \ldots, a_n$, with a_n a tuple of realizations of partial types in Φ over some parameters A, such that

(1) $a_n \not\mathop{\smile\!\!\!\!\!\!\perp}_{\Sigma\mathrm{cl}(A)} a_0$;
(2) $a_{i+1} \mathop{\smile\!\!\!\!\!\!\perp}_{\Sigma\mathrm{cl}(Aa_i)} a_0 \ldots a_{i-1}$ for $1 \le i < n$;
(3) $\Sigma\mathrm{cl}(Aa_0 \ldots a_{i-1}a_i) \cap \Sigma\mathrm{cl}(Aa_0 \ldots a_{i-1}a_{i+1}) = \Sigma\mathrm{cl}(Aa_0 \ldots a_{i-1})$ for $0 \le i < n$.

Remark 4.2. Pillay [20] requires $a_n \mathop{\smile\!\!\!\!\!\!\perp}_{Aa_i} a_0 \ldots a_{i-1}$ for $1 \le i < n$ in item (2). We follow the variant proposed by Evans and Nübling [6] which seems more natural and which implies

$$a_n \ldots a_{i+1} \mathop{\smile\!\!\!\!\!\!\perp}_{\Sigma\mathrm{cl}(Aa_i)} a_0 \ldots a_{i-1}.$$

Lemma 4.3. *If* Σ' *is a* Σ*-analysable family of partial types, then* n-Σ-*ample implies* n-Σ'-*ample, and in particular* n-*ample.*

Proof: As in [20, Remark 3.7] we replace a_i by

$$a_i' = \mathrm{Cb}(a_n' \ldots a_{i+1}'/\Sigma\mathrm{cl}(Aa_i))$$

for $i < n$, where $a_n' = a_n$. Then

$$a_n' \ldots a_{i+1}' \mathop{\smile\!\!\!\!\!\!\perp}_{a_i'} \Sigma\mathrm{cl}(Aa_i) \quad \text{and} \quad a_n' \ldots a_{i+1}' \mathop{\smile\!\!\!\!\!\!\perp}_{\Sigma\mathrm{cl}(Aa_i)} \Sigma\mathrm{cl}(Aa_0 \ldots a_i)$$

by Fact 3.18, whence

$$a_n' \ldots a_{i+1}' \mathop{\smile\!\!\!\!\!\!\perp}_{a_i'} \Sigma\mathrm{cl}(Aa_0 \ldots a_i).$$

Put $A' = \Sigma\mathrm{cl}(A)\cap\mathrm{bdd}(Aa_0')$. Then $A \subseteq A' \subseteq \Sigma\mathrm{cl}(A)$, whence $\Sigma\mathrm{cl}(A) = \Sigma\mathrm{cl}(A')$, and $a_0' \mathop{\smile\!\!\!\!\!\!\;|}_{A'} \Sigma\mathrm{cl}(A)$. Now $a_n \not\mathop{\smile\!\!\!\!\!\!\;|}_{\Sigma\mathrm{cl}(A')} a_0$ implies $a_n' \not\mathop{\smile\!\!\!\!\!\!\;|}_{\Sigma\mathrm{cl}(A)} a_0'$, whence $a_n' \not\mathop{\smile\!\!\!\!\!\!\;|}_{\Sigma'\mathrm{cl}(A)} a_0'$. Clearly $a_{i+1}' \mathop{\smile\!\!\!\!\!\!\;|}_{a_i'} \Sigma\mathrm{cl}(Aa_0\dots a_i)$ implies

$$a_{i+1}' \mathop{\smile\!\!\!\!\!\!\;|}_{\Sigma'\mathrm{cl}(A'a_i')} a_0'\dots a_{i-1}'$$

for $1 \leq i < n$. Finally,

$$A'a_0'\dots a_i'a_{i+1}' \mathop{\smile\!\!\!\!\!\!\;|}_{\Sigma'\mathrm{cl}(A'a_0'\dots a_{i-1}')} \Sigma\mathrm{cl}(Aa_0\dots a_{i-1})$$

yields

$$\Sigma'\mathrm{cl}(A'a_0'\dots a_{i-1}'a_{i+1}'), \Sigma'\mathrm{cl}(A'a_0'\dots a_i') \mathop{\smile\!\!\!\!\!\!\;|}_{\Sigma'\mathrm{cl}(A'a_0'\dots a_{i-1}')} \Sigma\mathrm{cl}(Aa_0\dots a_{i-1}),$$

so

$$\Sigma'\mathrm{cl}(A'a_0'\dots a_{i-1}'a_{i+1}') \cap \Sigma'\mathrm{cl}(A'a_0'\dots a_i') \subseteq \Sigma\mathrm{cl}(Aa_0\dots a_{i-1})$$

implies

$$\Sigma'\mathrm{cl}(A'a_0'\dots a_{i-1}'a_{i+1}') \cap \Sigma'\mathrm{cl}(A'a_0'\dots a_i') \subseteq \Sigma'\mathrm{cl}(A'a_0'\dots a_{i-1}').$$

□

This also shows that in Definition 4.1 one may require $a_0, \dots, a_{n-1}$ to lie in Φ^{heq}, and $a_{i+1} \mathop{\smile\!\!\!\!\!\!\;|}_{a_i} \Sigma\mathrm{cl}(Aa_0\dots a_i)$.

Remark 4.4. [20, Lemma 3.2 and Corollary 3.3] If $a_0, \dots, a_n$ witness n-Σ-ampleness over A, then $a_n \not\mathop{\smile\!\!\!\!\!\!\;|}_{\Sigma\mathrm{cl}(Aa_0\dots a_{i-1})} a_i$ for all $i < n$. Hence $a_i, \dots, a_n$ witness $(n-i)$-Σ-ampleness over $Aa_0\dots a_{i-1}$. Thus n-Σ-ample implies i-Σ-ample for all $i \leq n$.

Remark 4.5. It is clear from the definition that even though Φ might be a complete type p, if p is not n-Σ-ample, neither is any extension of p, not only the non-forking ones.

For $n = 1$ and $n = 2$ there are alternative definitions of non-n-Σ-ampleness:

Definition 4.6.
(1) Φ is *Σ-based* if $\mathrm{Cb}(a/\Sigma\mathrm{cl}(B)) \subseteq \Sigma\mathrm{cl}(aA)$ for any tuple a of realizations of partial types in Φ over some parameters A and any $B \supseteq A$.
(2) Φ is *Σ-CM-trivial* if $\mathrm{Cb}(a/\Sigma\mathrm{cl}(AB)) \subseteq \Sigma\mathrm{cl}(A, \mathrm{Cb}(a/\Sigma\mathrm{cl}(AC)))$ for any tuple a of realizations of partial types in Φ over some parameters A and any $B \subseteq C$ such that $\Sigma\mathrm{cl}(ABa)\cap\Sigma\mathrm{cl}(AC) = \Sigma\mathrm{cl}(AB)$.

Lemma 4.7.
(1) *Φ is Σ-based if and only if Φ is not 1-Σ-ample.*

(2) Φ *is* Σ*-CM-trivial if and only if* Φ *is not* 2-Σ*-ample.*

Proof: (1) Suppose Φ is Σ-based and consider a_0, a_1, A with $\Sigma\mathrm{cl}(Aa_0) \cap \Sigma\mathrm{cl}(Aa_1) = \Sigma\mathrm{cl}(A)$. Put $a = a_1$ and $B = Aa_0$. By Σ-basedness

$$\mathrm{Cb}(a/\Sigma\mathrm{cl}(B)) \subseteq \Sigma\mathrm{cl}(Aa) \cap \Sigma\mathrm{cl}(B) = \Sigma\mathrm{cl}(A).$$

Hence $a \mathop{\smile\!\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(A)} \Sigma\mathrm{cl}(B)$, whence $a_1 \mathop{\smile\!\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(A)} a_0$, so Φ is not 1-Σ-ample.

Conversely, if Φ is not Σ-based, let a, A, B be a counterexample. Put $a_0 = \mathrm{Cb}(a_1/\Sigma\mathrm{cl}(B))$ and $a_1 = a$. Then $a_0 \notin \Sigma\mathrm{cl}(Aa_1)$. Now take

$$A' = \mathrm{bdd}(Aa_0a_1) \cap \Sigma\mathrm{cl}(Aa_0) \cap \Sigma\mathrm{cl}(Aa_1).$$

Then $\Sigma\mathrm{cl}(A'a_0) \cap \Sigma\mathrm{cl}(A'a_1) = \Sigma\mathrm{cl}(A')$ by Lemma 3.21.

Suppose $a_1 \mathop{\smile\!\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(A')} a_0$. Since $\Sigma\mathrm{cl}(A') \subseteq \Sigma\mathrm{cl}(Aa_0) \subseteq \Sigma\mathrm{cl}(B)$ we have $a_1 \mathop{\smile\!\!\!\!\!\!\mid}_{a_0} \Sigma\mathrm{cl}(A')$. As $a_0 = \mathrm{Cb}(a_1/\Sigma\mathrm{cl}(B))$, this implies

$$a_0 \subseteq \Sigma\mathrm{cl}(A') \subseteq \Sigma\mathrm{cl}(Aa_1),$$

a contradiction. Hence a_0, a_1, A' witness 1-Σ-ampleness of Φ.

(2) Suppose Φ is Σ-CM-trivial and consider a_0, a_1, a_2, A with

$$\begin{gathered}\Sigma\mathrm{cl}(Aa_0) \cap \Sigma\mathrm{cl}(Aa_1) = \Sigma\mathrm{cl}(A),\\ \Sigma\mathrm{cl}(Aa_0a_1) \cap \Sigma\mathrm{cl}(Aa_0a_2) = \Sigma\mathrm{cl}(Aa_0), \quad \text{and}\\ a_2 \mathop{\smile\!\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(Aa_1)} a_0.\end{gathered}$$

Put $a = a_2$, $B = a_0$ and $C = a_0a_1$. Then

$$a_2 \mathop{\smile\!\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(Aa_1)} \Sigma\mathrm{cl}(Aa_0a_1),$$

so $\mathrm{Cb}(a/\Sigma\mathrm{cl}(AC)) \subseteq \Sigma\mathrm{cl}(Aa_1)$. Moreover

$$\Sigma\mathrm{cl}(ABa) \cap \Sigma\mathrm{cl}(AC) = \Sigma\mathrm{cl}(AB),$$

whence by Σ-CM-triviality

$$\begin{aligned}\mathrm{Cb}(a/\Sigma\mathrm{cl}(AB)) &\subseteq \Sigma\mathrm{cl}(A, \mathrm{Cb}(a/AC)) \cap \Sigma\mathrm{cl}(AB)\\ &\subseteq \Sigma\mathrm{cl}(Aa_1) \cap \Sigma\mathrm{cl}(Aa_0) = \Sigma\mathrm{cl}(A).\end{aligned}$$

Hence $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(A)} a_0$, so Φ is not 2-Σ-ample.

Conversely, if Φ is not Σ-CM-trivial, let a, A, B, C be a counterexample. Put

$$a_0 = \mathrm{Cb}(a/\Sigma\mathrm{cl}(AB)), \quad a_1 = \mathrm{Cb}(a/\Sigma\mathrm{cl}(AC)), \quad a_2 = a,$$

$$A' = \mathrm{bdd}(Aa_0a_1) \cap \Sigma\mathrm{cl}(Aa_0) \cap \Sigma\mathrm{cl}(Aa_1) \subseteq \Sigma\mathrm{cl}(AB).$$

Then $a_2 \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma\mathrm{cl}(A'a_1)} a_0$ and $a_0 \notin \Sigma\mathrm{cl}(Aa_1)$; by Lemma 3.21
$$\Sigma\mathrm{cl}(A'a_0) \cap \Sigma\mathrm{cl}(A'a_1) = \Sigma\mathrm{cl}(A').$$
Moreover, $a_2 \mathop{\smile\!\!\!\!\!\!\,|}_{a_0} \Sigma\mathrm{cl}(AB)$ implies
$$\Sigma\mathrm{cl}(A'a_0a_2) \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma\mathrm{cl}(A'a_0)} \Sigma\mathrm{cl}(AB).$$
Thus
$$\begin{aligned}\Sigma\mathrm{cl}(A'a_0a_2) \cap \Sigma\mathrm{cl}(A'a_0a_1) &\subseteq \Sigma\mathrm{cl}(ABa) \cap \Sigma\mathrm{cl}(AC)\\ &= \Sigma\mathrm{cl}(AB) \cap \Sigma\mathrm{cl}(A'a_0a_2) = \Sigma\mathrm{cl}(A'a_0).\end{aligned}$$
Suppose $a_2 \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma\mathrm{cl}(A')} a_0$. Since $a_2 \mathop{\smile\!\!\!\!\!\!\,|}_{a_0} \Sigma\mathrm{cl}(A')$ we obtain
$$a_0 = \mathrm{Cb}(a/\Sigma\mathrm{cl}(AB)) = \mathrm{Cb}(a/a_0\Sigma\mathrm{cl}(A')) \subseteq \Sigma\mathrm{cl}(A') \subseteq \Sigma\mathrm{cl}(Aa_1),$$
a contradiction. Hence a_0, a_1, a_2, A' witness 2-Σ-ampleness of Φ. □

In our definition of Σ-ampleness, we only consider the type of a_n over a Σ-closed set, namely $\Sigma\mathrm{cl}(A)$. This seems natural since the idea of Σ-closure is to work *modulo* Σ. However, sometimes one needs a stronger notion which takes care of all types. Let us first look at $n = 1$ and $n = 2$.

Definition 4.8.
- Φ is *strongly Σ-based* if $\mathrm{Cb}(a/B) \subseteq \Sigma\mathrm{cl}(aA)$ for any tuple a of realizations of partial types in Φ over some A and any $B \supseteq A$.
- Φ is *strongly Σ-CM-trivial* if $\mathrm{Cb}(a/AB) \subseteq \Sigma\mathrm{cl}(A, \mathrm{Cb}(a/AC)$ for any tuple a of realizations of partial types in Φ over some A and any $B \subseteq C$ with $\Sigma\mathrm{cl}(ABa) \cap \mathrm{bdd}(AC) = \mathrm{bdd}(AB)$.

Remark 4.9. $\mathrm{Cb}(a/\Sigma\mathrm{cl}(B)) \subseteq \mathrm{bdd}(\mathrm{Cb}(a/B), a) \cap \Sigma\mathrm{cl}(\mathrm{Cb}(a/B))$.

Proof: By Fact 3.19 the independence $a \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{Cb}(a/B)} B$ implies
$$a \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{dcl}(a,\mathrm{Cb}(a/b)) \cap \Sigma\mathrm{cl}(\mathrm{Cb}(a/B))} \Sigma\mathrm{cl}(B).$$
The result follows. □

Conjecture. $\mathrm{Cb}(a/B) \subseteq \Sigma\mathrm{cl}(\mathrm{Cb}(a/\Sigma\mathrm{cl}(B)))$.

If this conjecture were true, strong and ordinary Σ-basedness and Σ-CM-triviality would obviously coincide. Since we have not been able to show this, we weaken our definition of ampleness.

Definition 4.10. Φ is *weakly n-Σ-ample* if there are tuples $a_0, \ldots, a_n$, where a_n is a tuple of realizations of partial types in Φ over A, with

(1) $a_n \not\perp_A a_0$.
(2) $a_{i+1} \perp_{Aa_i} a_0 \dots a_{i-1}$ for $1 \leq i < n$.
(3) $\mathrm{bdd}(Aa_0 \dots a_{i-1}a_i) \cap \Sigma\mathrm{cl}(Aa_0 \dots a_{i-1}a_{i+1}) = \mathrm{bdd}(Aa_0 \dots a_{i-1})$ for $i < n$.

Note that (3) implies that $\mathrm{tp}(a_i/Aa_0 \dots a_{i-1})$ is foreign to Σ by Fact 3.18 for all $i < n$, and so is $\mathrm{tp}(a_i/Aa_{i-1})$ by (2). If Σ is the family of bounded partial types, then weak and ordinary n-Σ-ampleness just equal n-ampleness.

Lemma 4.11. *An n-Σ-ample family of types is weakly n-Σ-ample. If Σ' is Σ-analysable, then a weakly n-Σ-ample family is weakly n-Σ'-ample, and in particular n-ample.*

Proof: If $a_0, \dots, a_n$ witness n-Σ-ampleness over A, we put $a'_n = a_n$,

$$a'_i = \mathrm{Cb}(a'_n \dots a'_{i+1}/\Sigma\mathrm{cl}(Aa_i)) \subseteq \Sigma\mathrm{cl}(Aa_i) \quad \text{for } n > i$$

and

$$A' = \mathrm{bdd}(Aa'_0) \cap \Sigma\mathrm{cl}(Aa'_1) \subseteq \Sigma\mathrm{cl}(Aa_0) \cap \Sigma\mathrm{cl}(Aa_1) = \Sigma\mathrm{cl}(A).$$

As in Lemma 4.3 we have for $i < n$

$$a'_n \dots a'_{i+1} \mathop{\perp}_{a'_i} \Sigma\mathrm{cl}(Aa_0 \dots a_i).$$

For $0 < i < n$ we obtain $a'_{i+1} \perp_{A'a'_i} a'_0 \dots a'_{i-1}$; moreover

$$\begin{aligned}
\mathrm{bdd}(A'a'_0 \dots a'_{i-1}a'_i) &\cap \Sigma\mathrm{cl}(A'a'_0 \dots a'_{i-1}a'_{i+1}) \\
&\subseteq \Sigma\mathrm{cl}(A'a'_0 \dots a'_{i-1}a'_i) \cap \Sigma\mathrm{cl}(A'a'_0 \dots a'_{i-1}a'_{i+1}) \\
&\subseteq \Sigma\mathrm{cl}(Aa_0 \dots a_{i-1}a_i) \cap \Sigma\mathrm{cl}(Aa_0 \dots a_{i-1}a_{i+1}) \\
&= \Sigma\mathrm{cl}(Aa_0 \dots a_{i-1}).
\end{aligned}$$

But then $a'_i \perp_{A'a'_0 \dots a'_{i-1}} \Sigma\mathrm{cl}(Aa_0 \dots a_{i-1})$ yields

$$\mathrm{bdd}(A'a'_0 \dots a'_{i-1}a'_i) \cap \Sigma\mathrm{cl}(A'a'_0 \dots a'_{i-1}a'_{i+1}) = \mathrm{bdd}(A'a'_0 \dots a'_{i-1}),$$

while $\mathrm{bdd}(A'a'_0) \cap \Sigma\mathrm{cl}(A'a'_1) = \mathrm{bdd}(A')$ follows from the definition of A'. Finally $a_n \not\perp_{\Sigma\mathrm{cl}(A)} a_0$ implies $a'_n \not\perp_{\Sigma\mathrm{cl}(A)} a'_0$, whence $a'_n \not\perp_{A'} a'_0$ as $\mathrm{tp}(a'_0/A')$ is foreign to Σ and $\Sigma\mathrm{cl}(A) = \Sigma\mathrm{cl}(A')$.

The second assertion is clear, since $\Sigma'\mathrm{cl}(A) \subseteq \Sigma\mathrm{cl}(A)$ for any set A. $\square$

This also shows that in Definition 4.10 one may require $a_0, \dots, a_{n-1}$ to lie in Φ^{heq}.

Lemma 4.12. (1) *Φ is strongly Σ-based iff Φ is not weakly* 1-Σ-*ample.*
(2) *Φ is strongly Σ-CM-trivial iff Φ is not weakly* 2-Σ-*ample.*

Proof: This is similar to the proof of Lemma 4.7, so we shall be concise.

(1) Suppose Φ is strongly Σ-based and consider a_0, a_1, A with

$$\mathrm{bdd}(Aa_0) \cap \Sigma\mathrm{cl}(Aa_1) = \mathrm{bdd}(A).$$

Put $a = a_1$ and $B = Aa_0$. By strong Σ-basedness

$$\mathrm{Cb}(a/B) \subseteq \Sigma\mathrm{cl}(Aa) \cap \mathrm{bdd}(B) = \mathrm{bdd}(A),$$

whence $a_1 \mathop{\smile\!\!\!\!\!\!|}_A a_0$, so Φ is not weakly 1-Σ-ample.

Conversely, if Φ is not strongly Σ-based, let a, A, B be a counterexample. Put $a_0 = \mathrm{Cb}(a_1/B)$ and $a_1 = a$. Then $a_0 \notin \Sigma\mathrm{cl}(Aa_1)$. Now take $A' = \mathrm{bdd}(Aa_0) \cap \Sigma\mathrm{cl}(Aa_1)$. Clearly $A' = \mathrm{bdd}(A'a_0) \cap \Sigma\mathrm{cl}(A'a_1)$. Suppose $a_1 \mathop{\smile\!\!\!\!\!\!|}_{A'} a_0$. Since $a_0 = \mathrm{Cb}(a_1/B)$ implies $a_1 \mathop{\smile\!\!\!\!\!\!|}_{a_0} A'$, we obtain

$$a_0 \subseteq \mathrm{bdd}(A') \subseteq \Sigma\mathrm{cl}(Aa_1),$$

a contradiction. Hence a_0, a_1, A' witness weak 1-Σ-ampleness of Φ.

(2) Suppose Φ is strongly Σ-CM-trivial and consider a_0, a_1, a_2, A with

$$\mathrm{bdd}(Aa_0) \cap \Sigma\mathrm{cl}(Aa_1) = \mathrm{bdd}(A),$$
$$\mathrm{bdd}(Aa_0a_1) \cap \Sigma\mathrm{cl}(Aa_0a_2) = \mathrm{bdd}(Aa_0), \quad \text{and}$$
$$a_2 \mathop{\smile\!\!\!\!\!\!|}_{Aa_1} a_0.$$

Put $a = a_2$, $B = a_0$ and $C = a_0a_1$. Then $\mathrm{Cb}(a/AC) \subseteq \mathrm{bdd}(Aa_1)$. Moreover

$$\Sigma\mathrm{cl}(ABa) \cap \mathrm{bdd}(AC) = \mathrm{bdd}(AB),$$

whence by strong Σ-CM-triviality

$$\begin{aligned}\mathrm{Cb}(a/AB) &\subseteq \Sigma\mathrm{cl}(A, \mathrm{Cb}(a/AC)) \cap \mathrm{bdd}(AB)\\ &\subseteq \Sigma\mathrm{cl}(Aa_1) \cap \mathrm{bdd}(Aa_0) = \mathrm{bdd}(A).\end{aligned}$$

Hence $a_2 \mathop{\smile\!\!\!\!\!\!|}_A a_0$, so Φ is not 2-Σ-ample.

Conversely, if Φ is not strongly Σ-CM-trivial, let a, A, B, C be a counterexample. Put

$$a_0 = AB, \quad a_1 = \mathrm{Cb}(a/AC), \quad a_2 = a,$$
$$A' = \mathrm{bdd}(Aa_0) \cap \Sigma\mathrm{cl}(Aa_1).$$

Then $a_2 \mathop{\smile\!\!\!\!\!\!|}_{A'a_1} a_0$ and $\mathrm{Cb}(a_2/AB) \notin \Sigma\mathrm{cl}(Aa_1) = \Sigma\mathrm{cl}(A'a_1)$; moreover

$$\mathrm{bdd}(A'a_0) \cap \Sigma\mathrm{cl}(A'a_1) = \mathrm{bdd}(A').$$

Clearly

$$\begin{aligned}\Sigma\mathrm{cl}(A'a_0a_2)\cap\mathrm{bdd}(A'a_0a_1) &\subseteq \Sigma\mathrm{cl}(ABa)\cap\mathrm{bdd}(AC)\\ &= \mathrm{bdd}(AB)=\mathrm{bdd}(A'a_0).\end{aligned}$$

Suppose $a_2 \mathop{\smile\!\!\!\!\!\mid}_{A'} a_0$. Then $\mathrm{Cb}(a_2/AB)\in\mathrm{bdd}(A')\subseteq\Sigma\mathrm{cl}(Aa_1)$, a contradiction. Hence a_0,a_1,a_2,A' witness weak 2-Σ-ampleness of Φ. $\square$

Lemma 4.13. *If Φ is not (weakly) n-Σ-ample, neither is the family of $\emptyset$-conjugates of $\mathrm{tp}(a/A)$ for any $a\in\Sigma\mathrm{cl}(\bar aA)$, where $\bar a$ is a tuple of realizations of partial types in Φ over A.*

Proof: Suppose the family of $\emptyset$-conjugates of $\mathrm{tp}(a/A)$ is n-Σ-ample, as witnessed by $a_0,\dots,a_n$ over some parameters B. There is a tuple $\bar a$ of realizations of partial types in Φ over some $\emptyset$-conjugates of A inside B such that $a_n\in\Sigma\mathrm{cl}(\bar aB)$; we may choose it such that

$$\bar a \mathop{\smile\!\!\!\!\!\mid}_{a_nB} a_0\dots a_{n-1}.$$

Then $\bar a \mathop{\smile\!\!\!\!\!\mid}_{a_{n-1}a_nB} a_0\dots a_{n-2}$, and hence

$$\bar a \mathop{\smile\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(a_{n-1}a_nB)} a_0\dots a_{n-2}.$$

As $a_n \mathop{\smile\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(a_{n-1}B)} a_0\dots a_{n-2}$ implies

$$\Sigma\mathrm{cl}(a_{n-1}a_nB) \mathop{\smile\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(a_{n-1}B)} a_0\dots a_{n-2}$$

by Fact 3.19, we get

$$\bar a \mathop{\smile\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(a_{n-1}B)} a_0\dots a_{n-2}.$$

We also have $\bar a \mathop{\smile\!\!\!\!\!\mid}_{a_0\dots a_{n-2}a_nB} a_{n-1}$, whence

$$\Sigma\mathrm{cl}(a_0\dots a_{n-2}\bar aB) \mathop{\smile\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(a_0\dots a_{n-2}a_nB)} \Sigma\mathrm{cl}(a_0\dots a_{n-2}a_{n-1}B); \tag{1}$$

since Σ-closure is boundedly closed,

$$\begin{aligned}\Sigma\mathrm{cl}(a_0\dots a_{n-2}\bar aB)&\cap\Sigma\mathrm{cl}(a_0\dots a_{n-2}a_{n-1}B)\\ &\subseteq \Sigma\mathrm{cl}(a_0\dots a_{n-2}a_nB)\cap\Sigma\mathrm{cl}(a_0\dots a_{n-2}a_{n-1}B)\\ &= \Sigma\mathrm{cl}(a_0\dots a_{n-2}B).\end{aligned}$$

Finally, $\bar a \mathop{\smile\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(B)} a_0$ would imply $\Sigma\mathrm{cl}(\bar aB) \mathop{\smile\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(B)} a_0$ by Fact 3.19, and hence $a_n \mathop{\smile\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(B)} a_0$, a contradiction. Thus $\bar a \not\mathop{\smile\!\!\!\!\!\mid}_{\Sigma\mathrm{cl}(B)} a_0$, and $a_0,\dots,a_{n-1},\bar a$ witness n-Σ-ampleness of Φ over B, a contradiction.

Now suppose $a_0, \dots, a_n$ witness weak n-Σ-ampleness over B, and choose $\bar{a}$ as before. Then easily $\bar{a}a_n \mathop{\smile\!\!\!\!\!\!\,|}_{Ba_{n-1}} a_0 \dots a_{n-2}$, yielding (2) from the definition. Moreover, equation (1) implies

$$\begin{aligned}\Sigma\mathrm{cl}(a_0 \dots a_{n-2}\bar{a}B) &\cap \mathrm{bdd}(a_0 \dots a_{n-2}a_{n-1}B)\\ &\subseteq \Sigma\mathrm{cl}(a_0 \dots a_{n-2}a_nB) \cap \mathrm{bdd}(a_0 \dots a_{n-2}a_{n-1}B)\\ &= \mathrm{bdd}(a_0 \dots a_{n-2}B).\end{aligned}$$

Finally suppose $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_B a_0$. Since $\mathrm{tp}(a_0/B)$ is foreign to Σ, so is $\mathrm{tp}(a_0/B\bar{a})$. Then $a_0 \mathop{\smile\!\!\!\!\!\!\,|}_{B\bar{a}} \Sigma\mathrm{cl}(B\bar{a})$ by Fact 3.18, whence $a_0 \mathop{\smile\!\!\!\!\!\!\,|}_B a_n$, a contradiction. Thus $\bar{a} \not\mathop{\smile\!\!\!\!\!\!\,|}_B a_0$, and $a_0, \dots, a_{n-1}, \bar{a}$ witness weak n-Σ-ampleness of Φ over B, again a contradiction. $\square$

Lemma 4.14. *Suppose $B \mathop{\smile\!\!\!\!\!\!\,|}_A a_0 \dots a_n$. If $a_0, \dots, a_n$ witness (weak) n-Σ-ampleness over A, they witness (weak) n-Σ-ampleness over B.*

Proof: Clearly $B \mathop{\smile\!\!\!\!\!\!\,|}_{a_0 \dots a_{i-1}A} a_0 \dots a_{i+1}A$, so Lemma 3.20 yields

$$\Sigma\mathrm{cl}(Ba_0 \dots a_{i-1}a_i) \cap \Sigma\mathrm{cl}(Ba_0 \dots a_{i-1}a_{i+1}) = \Sigma\mathrm{cl}(Ba_0 \dots a_{i-1})$$

in the ordinary case, and

$$\mathrm{bdd}(Ba_0 \dots a_{i-1}a_i) \cap \Sigma\mathrm{cl}(Ba_0 \dots a_{i-1}a_{i+1}) = \mathrm{bdd}(Ba_0 \dots a_{i-1})$$

in the weak case, for all $i < n$.

Next, $a_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{Aa_0 \dots a_i} B$, whence $a_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma\mathrm{cl}(Aa_0 \dots a_i)} \Sigma\mathrm{cl}(Ba_i)$ by Lemma 3.19. Now $a_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma\mathrm{cl}(Aa_i)} a_0 \dots a_{i-1}$ implies $a_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma\mathrm{cl}(Aa_i)} \Sigma\mathrm{cl}(Aa_0 \dots a_i)$, whence

$$a_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma\mathrm{cl}(Ba_i)} a_0 \dots a_{i-1}$$

for $1 \leq i < n$ by transitivity. In the weak case, $a_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{Aa_i} a_0 \dots a_{i-1}$ implies $a_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{Aa_i} Ba_0 \dots a_{i-1}$ by transitivity, whence $a_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{Ba_i} a_0 \dots a_{i-1}$.

Finally, $a_n \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma\mathrm{cl}(A)} \Sigma\mathrm{cl}(B)$ by Fact 3.19, so $a_n \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma\mathrm{cl}(B)} a_0$ would imply $a_n \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma\mathrm{cl}(A)} a_0$, a contradiction. Hence $a_n \not\mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma\mathrm{cl}(B)} a_0$. In the weak case, $a_n \mathop{\smile\!\!\!\!\!\!\,|}_A B$ and $a_n \not\mathop{\smile\!\!\!\!\!\!\,|}_A a_0$ yield directly $a_n \not\mathop{\smile\!\!\!\!\!\!\,|}_B a_0$. $\square$

Lemma 4.15. *Let Ψ be an $\emptyset$-invariant family of types. If Φ and Ψ are not (weakly) n-Σ-ample, neither is $\Phi \cup \Psi$.*

Proof: Suppose $\Phi \cup \Psi$ is weakly n-Σ-ample, as witnessed by $a_0, \dots, a_n = bc$ over some parameters A, where b and c are tuples of realizations of partial types in Φ and Ψ, respectively. As Ψ is not n-Σ-ample, we must have $c \mathop{\smile\!\!\!\!\!\!|}_A a_0$. Put $a_0' = \mathrm{Cb}(bc/a_0A)$. Then $\mathrm{tp}(a_0'/A)$ is internal in $\mathrm{tp}(b/A)$ by Corollary 2.4. Put

$$a_n' = \mathrm{Cb}(a_0'/a_nA).$$

Then $\mathrm{tp}(a_n'/A)$ is $\mathrm{tp}(a_0'/A)$-internal and hence $\mathrm{tp}(b/A)$-internal. Note that $a_n \not\perp_A a_0$ implies $a_n \not\perp_A a_0'$, whence

$$a_n' \not\perp_A a_0' \quad \text{and} \quad a_n' \not\perp_A a_0.$$

Moreover $a_n' \in \mathrm{bdd}(Aa_n)$, so $a_0, \dots, a_{n-1}, a_n'$ witness weak n-Σ-ampleness over A.

As $\mathrm{tp}(a_n'/A)$ is $\mathrm{tp}(b/A)$-internal, there is $B \perp_A a_n'$ and a tuple $\bar b$ of realizations of $\mathrm{tp}(b/A)$ with $a_n' \in \mathrm{dcl}(B\bar b)$. We may assume

$$B \mathop{\perp}_{Aa_n'} a_0 \dots a_{n-1},$$

whence $B \perp_A a_0 \dots a_{n-1}a_n'$. Hence $a_0, \dots, a_{n-1}, a_n'$ witness weak n-Σ-ampleness over B by Lemma 4.14. As $a_n' \in \mathrm{dcl}(B\bar b)$, this contradicts non weak n-Σ-ampleness of Φ by Lemma 4.13.

The proof in the ordinary case is analogous, replacing A by $\Sigma\mathrm{cl}(A)$. □

Corollary 4.16. *For $i < \alpha$ let Φ_i be an $\emptyset$-invariant family of partial types. If Φ_i is not (weakly) n-Σ-ample for all $i < \alpha$, neither is $\bigcup_{i<\alpha} \Phi_i$.*

Proof: This just follows from the local character of forking and Lemma 4.15. □

Lemma 4.17. *If the family of $\emptyset$-conjugates of* $\mathrm{tp}(a/A)$ *is not (weakly) n-Σ-ample and $a \perp A$, then* $\mathrm{tp}(a)$ *is not (weakly) n-Σ-ample.*

Proof: Suppose $\mathrm{tp}(a)$ is (weakly) n-Σ-ample, as witnessed by $a_0, \dots, a_n$ over some parameters B, where $a_n = (b_i : i < k)$ is a tuple of realizations of $\mathrm{tp}(a)$. For each $i < k$ choose $B_i \perp_{b_i} (B, a_0 \dots a_n, B_j : j < i)$ with $B_ib_i \equiv Aa$. Then $B_i \perp b_i$, whence $(B_i : i < k) \perp Ba_0 \dots a_n$. Then $a_0, \dots, a_n$ witness (weak) n-Σ-ampleness over $(B, B_i : i < k)$ by Lemma 4.14, a contradiction, since $\mathrm{tp}(b_i/B_i)$ is an $\emptyset$-conjugate of $\mathrm{tp}(a/A)$ for all $i < k$. □

Remark 4.18. In fact, in the above Lemma it suffices to merely assume that the single type $\mathrm{tp}(a/A)$ is not (weakly) n-Σ-ample in the theory $T(A)$, using Corollary 4.16. It follows that ampleness is preserved under adding and forgetting parameters.

Corollary 4.19. *Let Ψ be an $\emptyset$-invariant family of types. If Ψ is Φ-internal and Φ is not (weakly) n-Σ-ample, neither is Ψ.*

Proof: Immediate from Lemmas 4.13 and 4.17. $\square$

Theorem 4.20. *Let Ψ be an $\emptyset$-invariant family of types. If Ψ is Φ-analysable and Φ is not (weakly) n-Σ-ample, neither is Ψ.*

Proof: Suppose Ψ is n-Σ-ample, as witnessed by $a_0, \ldots, a_n$ over some parameters A, where a_n is a tuple of realizations of Ψ. Put $a'_n = \ell_1^\Phi(a_n/\Sigma\mathrm{cl}(A) \cap \mathrm{bdd}(Aa_n))$. Then a_n and a'_n are domination-equivalent over $\Sigma\mathrm{cl}(A) \cap \mathrm{bdd}(Aa_n)$ by Theorem 3.6; moreover a_n and hence a'_n are independent of $\Sigma\mathrm{cl}(A)$ over $\Sigma\mathrm{cl}(A) \cap \mathrm{bdd}(Aa_n)$ by Fact 3.18, so a_n and a'_n are domination-equivalent over $\Sigma\mathrm{cl}(A)$. Thus $a_0, \ldots, a'_n$ witness non-Σ-ampleness over A, contradicting Corollary 4.19.

For the weak case we put $a'_n = \ell_1^\Phi(a_n/A)$. So a_n and a'_n are domination-equivalent over A, whence $a'_n \not\perp_A a_0$. Thus $a_0, \ldots, a'_n$ witness weak non-Σ-ampleness over A, contradicting again Corollary 4.19. $\square$

5. Analysability of canonical bases

As an immediate Corollary to Theorem 4.20, we obtain the following:

Theorem 5.1. *Suppose every type in T is non-orthogonal to a regular type, and let Σ be the family of all n-ample regular types. Then T is not weakly n-Σ-ample.*

Proof: A non n-ample type is not weakly Σ-ample by Lemma 4.11. So all regular types are not weakly n-Σ-ample. But every type is analysable in regular types by the non-orthogonality hypothesis. $\square$

Corollary 5.2. *Suppose every type in T is non-orthogonal to a regular type. Then $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ is analysable in the family of all non one-based regular types, for all a, b.*

Proof: This is just Theorem 5.1 for $n = 1$. $\square$

Note that a forking extension of a non one-based regular type of infinite rank may be one-based.

Remark 5.3. In fact, the proof shows more. For every type p let $\Sigma(p)$ be the collection of types in Σ not foreign to p. Then $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ is analysable in $\Sigma(\mathrm{tp}(\mathrm{Cb}(a/b)))$. In particular, if $\mathrm{tp}(a)$ or $\mathrm{tp}(b)$ has rank less than ω^α, so does $\mathrm{tp}(\mathrm{Cb}(a/b))$. Hence $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ is analysable in the family of all non one-based regular types of rank less than ω^α.

Corollary 5.2 is due to Zoé Chatzidakis for types of finite SU-rank in simple theories [5, Proposition 1.10]. In fact, she even obtains $\mathrm{tp}(\mathrm{Cb}(a/b)/\mathrm{bdd}(a) \cap \mathrm{bdd}(b))$ to be analysable in the family of non one-based types of rank 1, and to decompose in components, each of which is analysable in a non-orthogonality class of regular types. In infinite rank, one has to work modulo types of smaller rank. So let Σ_α be the collection of all partial types of SU-rank $< \omega^\alpha$, and $\mathcal{P}_\alpha$ be the family of non Σ_α-based types of SU-rank ω^α. Note that the Σ_α-based types of SU-rank ω^α are precisely the locally modular types of SU-rank ω^α.

Theorem 5.4. *Let* $b = \mathrm{Cb}(a/\Sigma_\alpha\mathrm{cl}(b))$ *be such that* $\mathrm{SU}(b) < \omega^{\alpha+1}$ *for some ordinal* α *and let* $A = \Sigma_\alpha\mathrm{cl}(b) \cap \Sigma_\alpha\mathrm{cl}(a)$. *Then* $\mathrm{tp}(b/A)$ *is* $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$*-analysable.*

Proof: Firstly, if $a \in \Sigma_\alpha\mathrm{cl}(b)$ then $a = b \in A$. Similarly, if $b \in \Sigma_\alpha\mathrm{cl}(a)$ then $b \in A$; in both cases $\mathrm{tp}(b/A)$ is trivially $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable. Hence we may assume $a \notin \Sigma_\alpha\mathrm{cl}(b)$ and $b \notin \Sigma_\alpha\mathrm{cl}(a)$.

Suppose towards a contradiction that the result is false and consider a counterexample a, b with $\mathrm{SU}(b)$ minimal modulo ω^α and then $\mathrm{SU}(b/\Sigma_\alpha\mathrm{cl}(a))$ being maximal modulo ω^α. Note that this implies

$$\omega^\alpha \leq \mathrm{SU}(b/a) \leq \mathrm{SU}(b/A) \leq \mathrm{SU}(b) < \omega^{\alpha+1}.$$

Clearly (after adding parameters) we may assume $A = \Sigma_\alpha\mathrm{cl}(\emptyset)$. Then for any c the type $\mathrm{tp}(c)$ is $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable iff $\mathrm{tp}(c/A)$ is.

Claim. *We may assume* $a = \mathrm{Cb}(b/\Sigma_\alpha\mathrm{cl}(a))$.

Proof of Claim: Put $\tilde{a} = \mathrm{Cb}(b/\Sigma_\alpha\mathrm{cl}(a))$ and $\tilde{b} = \mathrm{Cb}(\tilde{a}/\Sigma_\alpha\mathrm{cl}(b))$. Then $\tilde{a} \in \Sigma_\alpha\mathrm{cl}(a)$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_{\tilde{a}} b$. Hence $\Sigma_\alpha\mathrm{cl}(b) = \Sigma_\alpha\mathrm{cl}(\tilde{b})$ by [25, Lemma 3.5.8], and $\mathrm{tp}(\tilde{b})$ is not $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable either. Thus the pair $\tilde{a}, \tilde{b}$ also forms a counterexample. Moreover, $\mathrm{SU}(b)$ equals $\mathrm{SU}(\tilde{b})$ modulo ω^α and $\mathrm{SU}(b/\Sigma_\alpha\mathrm{cl}(a)) = \mathrm{SU}(b/\Sigma_\alpha\mathrm{cl}(\tilde{a}))$ equals $\mathrm{SU}(\tilde{b}/\Sigma_\alpha\mathrm{cl}(\tilde{a}))$ modulo ω^α. $\square$

Since a is definable over a finite part of a Morley sequence in $\mathrm{Lstp}(b/a)$ by supersimplicity of $\mathrm{tp}(b)$, we see that $\mathrm{SU}(a) < \omega^{\alpha+1}$. On the other hand, $a \notin \Sigma_\alpha\mathrm{cl}(b)$ implies $\mathrm{SU}(a/b) \geq \omega^\alpha$.

Let $\hat{a} \subseteq \mathrm{bdd}(a)$ and $\hat{b} \subseteq \mathrm{bdd}(b)$ be maximal $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable. Then $a \notin \Sigma_\alpha\mathrm{cl}(\hat{a})$ and $b \notin \Sigma_\alpha\mathrm{cl}(\hat{b})$, and $\mathrm{tp}(a/\hat{a})$ and $\mathrm{tp}(b/\hat{b})$ are foreign to $\Sigma_\alpha \cup \mathcal{P}_\alpha$. Since $\mathrm{Cb}(\hat{a}/b)$ and $\mathrm{Cb}(\hat{b}/a)$ are $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable, we obtain

$$a \mathop{\smile\!\!\!\!\!\!|}_{\hat a} \hat b \quad \text{and} \quad b \mathop{\smile\!\!\!\!\!\!|}_{\hat b} \hat a.$$

Claim. $\mathrm{tp}(b/\hat b)$ *and* $\mathrm{tp}(a/\hat a)$ *are both* Σ_α*-based.*

Proof of Claim: Let Φ be the family of Σ_α-based types of SU-rank ω^α. Then $\mathrm{tp}(a/\hat a)$ is $(\Sigma_\alpha \cup \mathcal{P}_\alpha \cup \Phi)$-analysable, but foreign to $\Sigma_\alpha \cup \mathcal{P}_\alpha$. Put $a_0 = \ell_1^\Phi(a/\hat a)$ and $b_0 = \ell_1^\Phi(b/\hat b)$. Then $a \,\underline{\square}_{\hat a}\, a_0$ and $b \,\underline{\square}_{\hat b}\, b_0$ by Lemma 3.6(3); as $a \mathop{\smile\!\!\!\!\!\!|}_{\hat a} \hat b$ and $b \mathop{\smile\!\!\!\!\!\!|}_{\hat b} \hat a$ we even have $a \,\underline{\square}_{\hat a\hat b}\, a_0$ and $b \,\underline{\square}_{\hat a\hat b}\, b_0$. Since $a \not\mathop{\smile\!\!\!\!\!\!|}_{\hat a\hat b} b$ we obtain $a_0 \not\mathop{\smile\!\!\!\!\!\!|}_{\hat a\hat b} b_0$. Moreover, $\mathrm{tp}(a_0/\hat a)$ and $\mathrm{tp}(b_0/\hat b)$ are Σ_α-based by Theorem 4.20 (or [26, Theorem 11]).

On the other hand, as $a_0 \not\mathop{\smile\!\!\!\!\!\!|}_{\hat b} b_0$, we see that $b' = \mathrm{Cb}(a_0/\Sigma_\alpha\mathrm{cl}(b_0))$ is not contained in $\hat b$ and hence is not $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable. So a_0, b' is another counterexample; by minimality of SU-rank b and b' have the same SU-rank modulo ω^α, whence $b \in \Sigma_\alpha\mathrm{cl}(b_0)$. Hence $\mathrm{tp}(b/\hat b)$ is Σ_α-based, as is $\mathrm{tp}(a/\hat a)$ since $a = \mathrm{Cb}(b/a)$ and $a \mathop{\smile\!\!\!\!\!\!|}_{\hat a} \hat b$. $\square$

Claim. $\Sigma_\alpha\mathrm{cl}(a, \hat b) = \Sigma_\alpha\mathrm{cl}(b, \hat a) = \Sigma_\alpha\mathrm{cl}(a, b)$.

Proof of Claim: As $\mathrm{tp}(a/\hat a)$ is Σ_α-based, we have

$$a \mathop{\smile\!\!\!\!\!\!|}_{\Sigma_\alpha\mathrm{cl}(a)\cap\Sigma_\alpha\mathrm{cl}(\hat a b)} \hat a b,$$

whence

$$\Sigma_\alpha\mathrm{cl}(a) \mathop{\smile\!\!\!\!\!\!|}_{\Sigma_\alpha\mathrm{cl}(a)\cap\Sigma_\alpha\mathrm{cl}(\hat a b)} b$$

by Fact 3.19. Thus $a = \mathrm{Cb}(b/\Sigma_\alpha\mathrm{cl}(a)) \in \Sigma_\alpha\mathrm{cl}(\hat a b)$. Similarly $b \in \Sigma_\alpha\mathrm{cl}(\hat b a)$. $\square$

Let now $(b)^\frown(b_j : j < \omega)$ be a Morley sequence in $\mathrm{tp}(b/a)$ and let $\hat b_j$ represent the part of b_j corresponding to $\hat b$. Then $(\hat b_j : j < \omega)$ is a Morley sequence in $\mathrm{tp}(\hat b/\hat a)$ since $a \mathop{\smile\!\!\!\!\!\!|}_{\hat a} \hat b$. As $\mathrm{SU}(\hat b) < \infty$ there is some minimal $m < \omega$ such that $\hat a = \mathrm{Cb}(\hat b/\hat a) \in \Sigma_\alpha\mathrm{cl}(\hat b, \hat b_j : j < m)$. Then $m > 0$, as otherwise $\Sigma_\alpha\mathrm{cl}(b) = \Sigma_\alpha\mathrm{cl}(\hat a, b) \ni a$, which is impossible. Moreover, $a \in \Sigma_\alpha\mathrm{cl}(\hat a, b_j)$ for all $j < m$ by invariance and hence, $a \in \Sigma_\alpha\mathrm{cl}(\hat b, b_j : j < m)$.

Put $b' = \mathrm{Cb}(b_j : j < m/\Sigma_\alpha\mathrm{cl}(b))$. Then $(b_j : j < m) \mathop{\smile\!\!\!\!\!\!|}_{b'\hat b} \Sigma_\alpha\mathrm{cl}(b)$, so by Fact 3.19

$$\Sigma_\alpha\mathrm{cl}(\hat b, b_j : j < m) \mathop{\smile\!\!\!\!\!\!|}_{\Sigma_\alpha\mathrm{cl}(b', \hat b)} \Sigma_\alpha\mathrm{cl}(b).$$

Then $a \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma_\alpha\mathrm{cl}(b',\hat b)} \Sigma_\alpha\mathrm{cl}(b)$, so $b = \mathrm{Cb}(a/\Sigma_\alpha\mathrm{cl}(b)) \in \Sigma_\alpha\mathrm{cl}(b', \hat b)$. As $b \notin \Sigma_\alpha\mathrm{cl}(\hat b)$ we obtain $b' \notin \Sigma_\alpha\mathrm{cl}(\hat b)$.

Claim. $\mathrm{tp}(b'/\Sigma_\alpha\mathrm{cl}(b') \cap \Sigma_\alpha\mathrm{cl}(b_j : j < m))$ *is not* $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$*-analysable.*

Proof of Claim: Note first that $(b_j : j < m) \mathop{\smile\!\!\!\!\!\!\,|}_a b$ implies

$$\Sigma_\alpha\mathrm{cl}(b_j : j < m) \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma_\alpha\mathrm{cl}(a)} \Sigma_\alpha\mathrm{cl}(b)$$

by Fact 3.19, whence

$$\Sigma_\alpha\mathrm{cl}(b') \cap \Sigma_\alpha\mathrm{cl}(b_j : j < m) \subseteq \Sigma_\alpha\mathrm{cl}(b) \cap \Sigma_\alpha\mathrm{cl}(a) = \Sigma_\alpha\mathrm{cl}(\emptyset).$$

As $b \in \Sigma_\alpha\mathrm{cl}(b', \hat b)$ and $\mathrm{tp}(b/\hat b)$ is not $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable, neither is $\mathrm{tp}(b'/\hat b)$, nor *a fortiori* $\mathrm{tp}(b'/\Sigma_\alpha\mathrm{cl}(\emptyset))$. $\square$

As $b' = \mathrm{Cb}(b_j : j < m/\Sigma_\alpha\mathrm{cl}(b'))$, the pair $(b_j : j < m), b'$ forms another counterexample. By minimality $\mathrm{SU}(b)$ equals $\mathrm{SU}(b')$ modulo ω^α, which implies $\Sigma_\alpha\mathrm{cl}(b) = \Sigma_\alpha\mathrm{cl}(b')$.

As $\mathrm{tp}(b_j/\hat b_j)$ is foreign to $\Sigma_\alpha \cup \mathcal{P}_\alpha$ and $\hat b$ is $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$-analysable, we obtain $\hat b \mathop{\smile\!\!\!\!\!\!\,|}_{(\hat b_j : j<m)} (b_j : j < m)$ and hence by Fact 3.19

$$\hat b \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma_\alpha\mathrm{cl}(\hat b_j : j<m)} \Sigma_\alpha\mathrm{cl}(b_j : j < m).$$

On the other hand, as $\hat a \in \Sigma_\alpha\mathrm{cl}(\hat b, \hat b_j : j < m)$ but $\hat a \notin \Sigma_\alpha\mathrm{cl}(\hat b_j : j < m)$ by minimality of m, we get

$$\mathrm{SU}(\hat b/\Sigma_\alpha\mathrm{cl}(\hat b_j : j < m) >_\alpha \mathrm{SU}(\hat b/\hat a, \Sigma_\alpha\mathrm{cl}(\hat b_j : j < m)),$$

where the index α indicates modulo ω^α.

Moreover, as $\hat b \mathop{\smile\!\!\!\!\!\!\,|}_{\hat a} a$ we get $\hat b \mathop{\smile\!\!\!\!\!\!\,|}_{\Sigma_\alpha\mathrm{cl}(\hat a)} \Sigma_\alpha\mathrm{cl}(a)$, i.e. $\mathrm{SU}(\hat b/\Sigma_\alpha\mathrm{cl}(\hat a)) = \mathrm{SU}(\hat b/\Sigma_\alpha\mathrm{cl}(a))$. Since $\Sigma_\alpha\mathrm{cl}(b) = \Sigma_\alpha\mathrm{cl}(b')$ and $b \in \Sigma_\alpha\mathrm{cl}(a\hat b)$ we obtain

$$\begin{aligned}\mathrm{SU}(b'/\Sigma_\alpha\mathrm{cl}(b_j : j < m)) &=_\alpha \mathrm{SU}(b/\Sigma_\alpha\mathrm{cl}(b_j : j < m))\\ &\geq_\alpha \mathrm{SU}(\hat b/\Sigma_\alpha\mathrm{cl}(b_j : j < m)) =_\alpha \mathrm{SU}(\hat b/\Sigma_\alpha\mathrm{cl}(\hat b_j : j < m))\\ &>_\alpha \mathrm{SU}(\hat b/\hat a, \Sigma_\alpha\mathrm{cl}(\hat b_j : j < m)) =_\alpha \mathrm{SU}(\hat b/\Sigma_\alpha\mathrm{cl}(\hat a))\\ &=_\alpha \mathrm{SU}(\hat b/\Sigma_\alpha\mathrm{cl}(a)) =_\alpha \mathrm{SU}(b/\Sigma_\alpha\mathrm{cl}(a)),\end{aligned}$$

contradicting the maximality of $\mathrm{SU}(b/\Sigma_\alpha\mathrm{cl}(a))$ modulo ω^α. This finishes the proof. $\square$

As a corollary we obtain Chatzidakis' Theorem for the finite SU-rank case, apart from the decomposition in orthogonal components:

Corollary 5.5. *Let* $b = \mathrm{bdd}(\mathrm{Cb}(a/b))$ *be such that* $\mathrm{SU}(b) < \omega$. *Then* $\mathrm{tp}(b/\mathrm{bdd}(b) \cap \mathrm{bdd}(a))$ *is analysable in the family of all non one-based types of* SU*-rank* 1*.*

6. Applications and the Canonical Base Property

In this section and the next, Σ^{nob} will be the family of non one-based regular types (seen as partial types). For the applications one would like (and often has) more than mere strongly Σ^{nob}-basedness of canonical bases:

Definition 6.1. A supersimple theory T has the *Canonical Base Property CBP* if $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ is almost Σ^{nob}-internal for all a, b.

Remark 6.2. In other words, in view of Corollary 5.2 a theory has the CBP if for all a, b the type $\mathrm{tp}(\mathrm{Cb}(a/b)/a)$ is Σ^{nob}-flat.

It had been conjectured that all supersimple theories of finite rank have the CBP, but Hrushovski has constructed a counter-example [14].

Remark 6.3. Chatzidakis has shown for types of finite SU-rank that the CBP implies that even $\mathrm{tp}(\mathrm{Cb}(a/b)/\mathrm{bdd}(a) \cap \mathrm{bdd}(b))$ is almost Σ^{nob}-internal [5, Theorem 1.15].

Example. The CBP holds for types of finite rank in

- Differentially closed fields in characteristic 0 [23].
- Generic difference fields [23, 5].
- Compact complex spaces [4, 7, 22].

Moreover, in those cases we have a good knowledge of the non one-based types.

Kowalski and Pillay [16, Section 4] have given some consequences of strongly Σ-basedness in the context of groups. In fact, they work in a theory with the CBP, but they remark that their results hold, with Σ-*analysable* instead of *almost* Σ*-internal*, in all stable strongly Σ-based theories.

Fact 6.4. *Let* G *be an* $\emptyset$*-hyperdefinable strongly* Σ*-based group in a stable theory.*

(1) *If* $H \leq G$ *is connected with canonical parameter* c*, then* $\mathrm{tp}(c)$ *is* Σ*-analysable.*
(2) $G/Z(G)$ *is* Σ*-analysable.*

An inspection of their proof shows that mere simplicity of the ambient theory is sufficient, replacing centers by approximate centers and connectivity by local connectivity. Recall that the *approximate center* of a group G is

$$\tilde{Z}(G) = \{g \in G : [G : C_G(g)] < \infty\}.$$

A subgroup $H \leq G$ is *locally connected* if for all group-theoretic or model-theoretic conjugates H^σ of H, if H and H^σ are commensurate, then $H = H^\sigma$. Locally connected subgroups and their cosets have canonical parameters; every subgroup is commensurable with a unique minimal locally connected subgroup, its *locally connected component.* For more details about the approximate notions, the reader is invited to consult [25, Definition 4.4.9 and Proposition 4.4.10].

Proposition 6.5. *Let G be an $\emptyset$-hyperdefinable strongly Σ-based group in a simple theory.*

(1) *If $H \leq G$ is locally connected with canonical parameter c, then* $\mathrm{tp}(c)$ *is Σ-analysable.*
(2) *$G/\tilde{Z}(G)$ is Σ-analysable.*

Proof: (1) Take $h \in H$ generic over c and $g \in G$ generic over c, h. Let d be the canonical parameter of gH. Then $\mathrm{tp}(gh/g, c)$ is the generic type of gH, so d is interbounded with $\mathrm{Cb}(gh/g, c)$. By strongly Σ-basedness, $\mathrm{tp}(d/gh)$ is Σ-analysable. But $c \in \mathrm{dcl}(d)$, so $\mathrm{tp}(c/gh)$ is Σ-analysable, as is $\mathrm{tp}(c)$ since $c \mathop{\smile\!\!\!\!\!\!\;|}\; gh$.

(2) For generic $g \in G$ put

$$H_g = \{(x, x^g) \in G \times G : x \in G\},$$

and let H_g^{lc} be the locally connected component of H_g. Then $g\tilde{Z}(G)$ is interbounded with the canonical parameter of H_g^{lc}, so $\mathrm{tp}(g\tilde{Z}(G))$ is Σ-analysable, as is $G/\tilde{Z}(G)$. □

Theorem 6.6. *Let G be an $\emptyset$-hyperdefinable strongly Σ-based group in a simple theory. If G is supersimple or type-definable, there is a normal nilpotent $\emptyset$-hyperdefinable subgroup N such that G/N is almost Σ-internal. In particular, a supersimple or type-definable group G in a simple theory has a normal nilpotent hyperdefinable subgroup N such that G/N is almost Σ^{nob}-internal.*

Proof: $G/\tilde{Z}(G)$ is Σ-analysable by Proposition 6.5. Hence there is a continuous sequence

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_\alpha \rhd \tilde{Z}(G)$$

of normal $\emptyset$-hyperdefinable subgroups such that successive quotients $Q_i = G_i/G_{i+1}$ are Σ-internal for all $i < \alpha$, and $G_\alpha/\tilde{Z}(G)$ is bounded.

Now G acts on every quotient Q_i. Let

$$N_i = \{g \in G : [Q_i : C_{Q_i}(g)] < \infty\}$$

be the approximate stabilizer of Q_i in G, again an $\emptyset$-hyperdefinable subgroup. If $(q_j : j < \kappa)$ is a long independent generic sequence in Q_i and g, g' are two elements of G which have the same action on all q_j for $j < \kappa$, there is some $j_0 < \kappa$ with $q_{j_0} \perp g, g'$. Since $g^{-1}g'$ stabilizes q_{j_0} it lies in N_i, and gN_i is determined by the sequence $(q_j, q_j^g : j < \kappa)$. Thus G/N_i is Q_i-internal, whence Σ-internal.

Put $N = \bigcap_{i<\alpha} N_i$. Since $\prod_{i<\alpha} G/N_i$ projects definably onto G/N, the latter quotient is also Σ-internal. In order to finish it now suffices to show that N is virtually nilpotent. In particular, we may assume that N is $\emptyset$-connected.

Consider the approximate ascending central series $\tilde{Z}_i(N)$. Note that N centralizes $G_\alpha/\tilde{Z}(G)$ by $\emptyset$-connectivity. Moreover, N approximatively stabilizes all quotients $(G_i \cap N)/(G_{i+1} \cap N)$. Hence, if $G_{i+1} \cap N \leq \tilde{Z}_j(N)$, then $G_i \cap N \leq \tilde{Z}_{j+1}(N)$. If G is supersimple, we may assume that all the Q_i are unbounded, so α is finite and $N = \tilde{Z}_{\alpha+2}(N)$. In the type-definable case, note that $\tilde{Z}_i(N)$ is relatively $\emptyset$-definable by [25, Lemma 4.2.6]. So by compactness the least ordinal α_i with $G_{\alpha_i} \cap N \leq \tilde{Z}_i(N)$ must be a successor ordinal, and $\alpha_{i+1} \leq \alpha_i - 1 < \alpha_i$. Hence the sequence must stop and there is $k < \omega$ with $N = \tilde{Z}_k(N)$. But then N is nilpotent by [25, Proposition 4.4.10.3]. $\square$

Remark 6.7. In a similar way one can show that if G acts definably and faithfully on a Σ-analysable group H and H is supersimple or type-definable, then there is a hyperdefinable normal nilpotent subgroup $N \triangleleft G$ such that G/N is almost Σ-internal.

7. Final Remarks

We have seen that for (weak) Σ-ampleness only the first level of an element is important. However, the difference between strong Σ^{nob}-basedness and the CBP is precisely the possible existence of a second (or higher) Σ^{nob}-level of $\mathrm{Cb}(a/b)$ over a, i.e. its non Σ^{nob}-flatness.

One might be tempted to try to prove the CBP replacing Σ^{nob}-closure by Σ_1^{nob}-closure. In fact it is possible to define a corresponding notion of Σ_1-ampleness, and to prove an analogue of Theorem 4.20. However, since Σ_1-closure is not a closure operator, the equivalence between Σ_1^{nob}-basedness (i.e. the CBP) and non 1-Σ_1^{nob}-ampleness breaks down. So far we have not found a way around this.

A possible approach to circumvent the failure of the CBP in general could be to use Theorem 6.6 in the applications, rather than establish the CBP for particular theories and use Fact 6.4 (or Proposition 6.5), but we have not looked into this.

Finally, it might be interesting to look for a variant of ampleness which does take all levels into account, as one might hope to obtain stronger structural consequences.

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Universitat de Barcelona; Departament de Lògica, Història i Filosofia de la Ciència, Montalegre 6, 08001 Barcelona, Spain

Current address: Université de Lyon; CNRS; Université Lyon 1; Institut Camille Jordan UMR5208, 43 bd du 11 novembre 1918, 69622 Villeurbanne Cedex, France

Université de Lyon; CNRS; Université Lyon 1; Institut Camille Jordan UMR5208, 43 bd du 11 novembre 1918, 69622 Villeurbanne Cedex, France

E-mail address: `palacin@math.univ-lyon1.fr`

E-mail address: `wagner@math.univ-lyon1.fr`